STATIONARY TEMPERATURE FIELD IN TRANSPIRATION COOLING

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A method is proposed for solving the two-dimensional and axisymmetric problems of the stationary temperature field of a porous body through whose pores a coolant flows. The proposed method is used to solve a number of typical problems associated with the cooling of porous bodies.

There are many engineering problems involving flows of gases and incompressible liquids in porous materials in the presence of large temperature drops at the boundaries of the flow region. These problems are primarily connected with atomic and chemical reactors, blast furnaces, etc. When it is necessary to provide reliable thermal protection for all or part of a structure operating under high-temperature conditions, it is quite effective to use porous materials cooled by a flow of gas or liquid through the pores to the hot surface. Transpiration cooling is most frequently used in aircraft and rocket engines, power engineering equipment, etc.

In the literature the problems of the analytical determination of the temperature field in the flow region are usually examined in the one-dimensional approximation [1–3].

Our object is to obtain an analytical solution for the steady-state temperature field under conditions of two-dimensional and axisymmetric flow of a "nonpolytropic" gas, i.e., a gas flowing at large temperature drops, an incompressible liquid, whose viscosity depends on temperature, and a hypothetical coolant, whose properties are invariant in the range of temperature variation investigated.

The problem is formulated as follows: a) the flow takes place in a nondeformable, uniformly porous medium and obeys the law

$$\mathrm{v} = -\frac{Kv^n}{\mu}\,\mathrm{grad}\,P, \tag{1}$$

where K is constant for a homogeneous porous medium; b) the temperatures of the porous body and the coolant are the same at any point of the flow region (this assumption may be considered justified if the internal surface of contact between the coolant and the body is sufficiently large. The results presented in [4,5] confirm the validity of this assumption); c) in the case of an incompressible liquid we assume that its heat capacity is constant. When a gas is used as the coolant it is assumed that it is a perfect gas satisfying the Clapeyron equation

$$P = \gamma RT.$$

Then the differential equation for the temperature of the porous body is written in the following form:

$$\Delta T - \frac{C_p\gamma}{\lambda}\,\mathrm{v}\,\mathrm{grad}\,T = 0. \tag{2}$$

The value of λ is assumed constant and equal to the mean value for the fluid and the porous body. Moreover, T must satisfy the corresponding boundary conditions.

In the case of axisymmetric flow it is possible to introduce the stream function V given by the expressions

$$r\gamma v_r = \frac{\partial V}{\partial z},$$

$$r\gamma v_z = -\frac{\partial V}{\partial r}. \tag{3}$$

Analogous expressions are written for the stream function in the case of two-dimensional flow.

It is easy to see that the surfaces P = const, V = = const, and φ = const are mutually orthogonal.

Instead of the cylindrical coordinates z, r, and φ, we introduce the orthogonal curvilinear coordinates $q_1 = P$, $q_2 = V$, and $q_3 = \varphi$. The Lamé coefficients for the new variables are determined from the equations

$$H_1 = \left|\frac{\partial \mathrm{r}}{\partial P}\right| = \frac{dS_p}{dP} = \frac{Kv^{n-1}}{\mu},$$

$$H_2 = \left|\frac{\partial \mathrm{r}}{\partial V}\right| = \frac{dS_V}{dV} = \frac{1}{\gamma v r},$$

$$H_3 = \left|\frac{\partial \mathrm{r}}{\partial \varphi}\right| = \frac{dS_\varphi}{d\varphi} = r.$$

Using expression (1), we transform Eq. (2) to the new coordinates:

$$\frac{\partial}{\partial P}\left(\chi\frac{\partial T}{\partial P}\right) + \frac{\partial}{\partial V}\left(\frac{r^2}{\chi}\frac{\partial T}{\partial V}\right) + \frac{C_p}{\lambda}\frac{\partial T}{\partial P} = 0, \tag{4}$$

where $\chi = \mu/K\gamma v^n$.

Similar calculations with transition to the curvilinear orthogonal coordinates P, V can also be made in the case of the two-dimensional problem. In the general case instead of Eq. (2) we obtain

$$\frac{\partial}{\partial P}\left(\chi\frac{\partial T}{\partial P}\right) + \frac{\partial}{\partial V}\left(\frac{r^{2\nu}}{\chi}\frac{\partial T}{\partial V}\right) + \frac{C_p}{\lambda}\frac{\partial T}{\partial P} = 0. \tag{5}$$

In the case of two-dimensional filtration $\nu = 0$ (here, r plays the role of the x coordinate of the rectangular Cartesian coordinate system xoy, and z the role of the y coordinate), while in the axisymmetric problem $\nu = 1$.

The solution of Eq. (5) must satisfy the boundary conditions for T(P,V) corresponding to the starting conditions for T(x, y) or T(z, r, φ).

We will consider several special cases of determination of the temperature field of practical interest, always assuming that the flow obeys Darcy's law ($n = 0$). In this case the expression for χ simplifies to

$$\chi = \frac{\mu}{K\gamma}.$$

Moreover, it is natural to assume that the boundary of the region in which the temperature distribution is sought is divided into two parts: a cold part—surface corresponding to coolant inlet—and a hot part—surface corresponding to coolant outlet.

1. The temperature and pressure on each of these parts of the boundary are constant (but different on each part).

Since γ is a constant quantity (incompressible liquid) or, for a compressible gas, satisfies the Clapeyron equation, while in both cases μ may be assumed to depend on temperature, for linear flow χ depends only on temperature and pressure:

$$\chi = \chi(P, T).$$

Therefore, in the given case it is possible to find a solution for T that depends only on P. Thus, Eq. (5) is transformed into a second-order ordinary differential equation in T:

$$\frac{d}{dP}\left(\chi\frac{dT}{dP}\right) + \frac{C_p}{\lambda}\frac{dT}{dP} = 0. \tag{6}$$

Integrating (6), we find

$$\chi\frac{dT}{dP} = -\frac{C_p}{\lambda}T + \text{const.} \tag{7}$$

Integrating (7), we have

a) for an incompressible liquid (γ = const),

$$\int \frac{\mu(T)}{B - ST}\,dT = P, \tag{8}$$

where $S = C_p\gamma K/\lambda$. We can integrate the left side of expression (8) if we have the relation $\mu = \mu(T)$ for the selected liquid.

b) for a compressible gas the temperature dependence of the gas viscosity can be approximately determined from the formula

$$\mu = \mu_0\left(\frac{T}{T_0}\right)^m.$$

Substituting in χ the value of γ from the Clapeyron equation, together with the expression for μ, and integrating expression (7) for $m = 1$, we obtain

$$P^2 = -\left[\frac{T}{4\Gamma^2 C_1} + \frac{T^2}{4\Gamma} + \frac{1}{8\Gamma^3 C_1^2}\ln(1 - 2\Gamma C_1 T)\right] + C_2, \tag{9}$$

where $\Gamma = C_p K T_0/4R\mu_0\lambda$.

The results obtained below remain essentially the same when the more general law $m \neq 1$ is considered.

In both case a) and case b) the constants of integration must be selected from the conditions at the boundary. In the particular case in question these conditions have the form

$$P = P_1,\ T = T_1,$$
$$P = P_2,\ T = T_2.$$

From these conditions we determine the values of the constants of integration in expressions (8) and (9).

In the flow region T (and hence P) can be determined as follows. Substituting in (2) the expression (1) for v and replacing μ and γ (in the case of a compressible gas) and also P, as indicated above, we obtain a differential equation for T. Replacing T by the new variable Θ from the expression

$$B - ST = \exp[\Theta], \tag{10}$$

in the case of an incompressible liquid, and

$$1 - 2\Gamma C_1 T = \exp[\Theta], \tag{10'}$$

in the case of a compressible gas, we arrive at the Laplace equation for the function Θ:

$$\Delta\Theta = 0. \tag{11}$$

Equation (11) must be solved with the boundary conditions for Θ corresponding to the starting conditions, using (10) or (10').

The results obtained can be extended to the case of arbitrary three-dimensional linear flow.

2. The pressure and temperature are constant at the cold surface, variable at the hot surface.

We assume that the coefficients μ and γ are constant and equal to the mean value of each coefficient in the given range of temperature variation. This assumption makes it possible to use the results of the theory of linear flow, in which it is shown that in the presence of a steady-state flow regime $P(x, y)$ is a harmonic function in the region considered [6]. Consequently, we can formulate and solve the Dirichlet problem for P.

In the case of the two-dimensional problem, taking the above assumption into account, we write Eq. (5) in the form

$$\frac{\partial^2 T}{\partial P^2} + \frac{K^2\gamma^2}{\mu^2}\frac{\partial^2 T}{\partial V^2} + \frac{C_p\gamma K}{\mu\lambda}\frac{\partial T}{\partial P} = 0.$$

We introduce the new variables

$$P = P,\ \psi = -\frac{\mu}{K\gamma}V.$$

In the coordinates P, ψ Eq. (5) takes the form

$$\frac{\partial^2 T}{\partial P^2} + \frac{\partial^2 T}{\partial\psi^2} + q\frac{\partial T}{\partial P} = 0, \tag{12}$$

where $q = C_p\gamma K/\mu\lambda$ = const.

By means of the substitution

$$T = U\exp\left[-\frac{q}{2}P\right]$$

Eq. (12) is reduced to the form

$$\frac{\partial^2 U}{\partial P^2} + \frac{\partial^2 U}{\partial \psi^2} - Q^2 U = 0. \qquad (12')$$

Here, $Q^2 = q^2/4$.

Let the simply connected flow region be symmetrical with respect to one of the axes, for example, ox, and let the boundary junction points also be symmetrical with respect to that axis. In this formulation of the problem the given flow region in the plane xoy will correspond to an infinite strip $R(\psi) \leq P \leq P_1$, $-\infty < \psi < +\infty$, in the plane $PO\psi$, since, as the junction point is approached, the quantity ψ increases without bound, taking infinitely large values at these points.

Thus, bearing in mind what was said above, we can now formulate the boundary value problem as follows: to find the solution of Eq. (12') within the infinite strip G: $-\infty < \psi < +\infty$, $R(\psi) \leq P \leq P_1$, satisfying the conditions

$$P = P_1,\ U_1 = 0,\ P = R(\psi),\ U_2 = \Phi(P, \psi). \qquad (13)$$

Here, $R(\psi)$ and $\Phi(P, \psi)$ are known functions corresponding to those given in the plane xoy.

From considerations of a physical nature, it follows that the temperature distribution will be symmetrical about the line $\psi = 0$ and consequently $(\partial U/\partial\psi)_{\psi=0} = 0$. Moreover, from the same considerations it is obvious that as $\psi \to \pm\infty$ and $U \to 0$, $\partial U/\partial\psi \to 0$.

Boundary value problem (12')–(13) can be reduced to a variational problem by selecting the functional for which the given equation would be the Euler equation and then investigating this functional for the extremum by the Ritz method. As is easy to verify, Eq. (12') is the Euler equation for the functional

$$I[U] = \int\limits_G\int \left[\left(\frac{dU}{\partial P}\right)^2 + \left(\frac{\partial U}{\partial \psi}\right)^2 + Q^2U^2\right] dP d\psi. \qquad (14)$$

In the Ritz method the values of I[U] are investigated not on arbitrary permissible curves of the given variational problem, but only on the linear combinations

$$U_k = \sum_{k=0}^{n} a_k \varphi_k(P, \psi),$$

where $\varphi_i|_D = 0$ $(i = 1, 2, \ldots, n)$; $\varphi_0|_D = \Phi(P, \psi)$; $a_0 \equiv 1$; D is the boundary of the region G; a_k are constant coefficients determined from the system of equations

$$\begin{cases} [\varphi_0, \varphi_1] + a_1[\varphi_1, \varphi_1] + \ldots + a_n[\varphi_n, \varphi_1] = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [\varphi_0, \varphi_n] + a_1[\varphi_1, \varphi_n] + \ldots + a_n[\varphi_n, \varphi_n] = 0. \end{cases} \qquad (15)$$

Here,

$$[\varphi_i, \varphi_j] = [\varphi_j, \varphi_i];$$

$$[\varphi_i, \varphi_j] = \int\limits_G\int \left[\frac{\partial\varphi_i}{\partial P}\frac{\partial\varphi_j}{\partial P} + \frac{\partial\varphi_i}{\partial\psi}\frac{\partial\varphi_j}{\partial\psi} + Q^2\varphi_i\varphi_j\right] dP d\psi.$$

As shown in [7], the coefficients $a_1 \ldots a_n$ are uniquely determined from system (15) if the coordinate functions are linearly independent.

As $\varphi_i(P, \psi)$ it is possible to take various combinations of trigonometric functions or polynomials. For example, for the first boundary value problem a strongly minimal coordinate system in an infinite strip on a two-dimensional plane can be written as follows [7]:

$$\frac{1}{\sqrt{1+\psi^2}} \sin i\pi \frac{P_1 - P}{P_1 - R(\psi)} \cos(2\, m \operatorname{arctg}\psi)$$

$$(i = 1, 2\ldots, m = 0, 1\ldots).$$

If $\varphi_i(P, \psi)$ is selected in the form

$$\frac{1}{\sqrt{1+\psi^2}} \sin i\pi \frac{P_1 - P}{P_1 - R(\psi)},$$

corresponding to m = 0 and satisfying the conditions $\partial\varphi_i/\partial\psi|_{\psi=0} = 0$, $\varphi_i|_D = 0$, then the expression for determining $[\varphi_i, \varphi_j]$ $(j \neq 0)$ takes the form

$$[\varphi_i, \varphi_j] = \left(\frac{ij\cos(j+i)\pi}{(j+i)^2} + \frac{ij\cos(i-j)\pi}{(i-j)^2}\right) \times \int_{-\infty}^{\infty} \frac{[R'(\psi)]^2}{[P_1 - R(\psi)](1+\psi^2)} d\psi.$$

Furthermore,

$$[\varphi_i, \varphi_0] = \int\limits_G\int \left[\frac{Q^2\Phi(P,\psi)}{\sqrt{1+\psi^2}} \sin i\pi \frac{P_1 - P}{P_1 - R(\psi)} - \frac{\partial\Phi}{\partial P} \frac{i\pi\cos i\pi \dfrac{P_1 - P}{P_1 - R(\psi)}}{[P_1 - R(\psi)]\sqrt{1+\psi^2}} - \frac{\partial\Phi}{\partial\psi} \times \left(\frac{\psi}{\sqrt{(1+\psi^2)^3}} \sin i\pi \frac{P_1 - P}{P_1 - R(\psi)} + \frac{(P_1 - P)R'(\psi)}{[P_1 - R(\psi)]^2\sqrt{1+\psi^2}} \times i\pi\cos i\pi \frac{P_1 - P}{P_1 - R(\psi)}\right)\right] dP d\psi.$$

NOTATION

v is the flow rate; P is the pressure; K is the permeability factor; μ is the coefficient of dynamic viscosity; n is the flow-rate exponent; when n = 0, the flow obeys Darcy's law; when n = −1, the flow is quadratic; T is the temperature; R is the gas constant; γ is the coolant density; C_p is the specific heat of the gas at constant pressure or the specific heat of an incompressible liquid; λ is the thermal conductivity; z, r, and φ are cylindrical coordinates; B, C_1, and C_2 are constants of integration; P_1 and T_1 are, respectively, the pressure and temperature at surface of body through which coolant is forced; P_2 and T_2 are, respectively, the pressure and temperature at the transpiration surface.

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